Bounds for binary codes of length less than 25

by

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Abstract

This paper presents improved bounds for A(n,d), the maximum number of codewords in a (linear or nonlinear) binary code of word length n and minimum distance d, and for A(n,d,w), the maximum number of binary vectors of length n, minimum distance d, and constant weight w, in the range $n \le 24$ and $d \le 10$. Some of the new values are A(9,4) = 20 (which was previously believed to follow from the results of Wax), A(13,6) = 32 (proving that the Nadler code is optimal), A(17,8) = 36 or 37, and A(21,8) = 512. The upper bounds on A(n,d) are found with help of linear programming, making use of the values of A(n,d,w).

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# 1. INTRODUCTION

The main purpose of this paper is to present tables of two of the most basic functions in coding theory, namely

> $A(n,d)$ = maximum number of codewords in any (linear or nonlinear) binary code of length n and minimum distance d between codewords,

and

> $A(n,d,w)$ = maximum number of codewords in any binary code of length n, constant weight w and minimum distance d,

in the range $n \le 24$, $d \le 10$. We also give a table of the function

> $T(w_1,n_1,w_2,n_2,d)$ = maximum number of codewords in a binary code of length $n_1+n_2$ and minimum distance d with exactly $w_1$ ones in the first $n_1$ coordinates and exactly $w_2$ ones in the last $n_2$ coordinates,

for $n_1+n_2 \le 24$, $d = 10$.*)

All of the upper bounds on $A(n,d)$ outside the Plotkin range $n \le 2d$ are obtained from modifications of Delsarte's linear programming method, making use of the values of $A(n,d,w)$ (see §3). The tables of $A(n,d,w)$ are important because they lead to bounds on $A(n,d)$, and in their own right for giving the size of the largest constant weight codes. They also give the solution to the following widely studied packing problem (ERDÖS & HANANI [17], KALBFLEISCH & STANTON [36], SCHÖNHEIM [51], STANTON, KALBFLEISCH & MULLIN [59]): What is $D(t,k,v)$, the maximum number of k-subsets of a v-set S, such

*) We would appreciate hearing of any improvements to the tables. (Send them for example to N.J.A. Sloane, Math. Research Center, Bell Labs, Murray Hill, N.J. 07974 U.S.A..)

that every t-subset of S is contained in at most one k-set? The answer is $D(t,k,v) = A(v,2k-2t+2,k)$, so that table 2 is also a table of values of $D(t,k,v)$.

Two recent papers which also use the linear programming approach are BEST & BROUWER [3] and McELIECE, RODEMICH, RUMSEY & WELCH [43].

Earlier tables of bounds on $A(n,d)$ were given in JOHNSON [33], McELIECE et al. [42] and SLOANE [53]. No table of $A(n,d,w)$ seems to have been published before, although unpublished tables of upper bounds exist (e.g. DELSARTE et al. [12], JOHNSON [32]). A table of $A(n,d,w)$ was promised in STANTON et al. [59] but has never appeared. A table of upper and lower bound on linear codes appeared in HELGERT & STINAFF [29].

The following notation is used in this paper. All codes are binary. An $(n,M,d)$ *code* consists of $M(\geq 1)$ binary vectors (called codewords) of length n such that any two codewords differ in at least d places, i.e. are at *(Hamming) distance* at least d apart. A code has *constant weight* w if each codeword contains w 1's, i.e. has *weight* w. An optimal code is a code with the maximum number of codewords for the given n and d (and for the given w, in the case of a constant weight code).

Let C be an $(n,M,d)$ code. The *weight distribution of* C *with respect to* a vector u is the $(n+1)$-tuple of integers $(A_i(u),\ i=0,\dots,n)$, where $A_i(u)$ is the number of codewords $v \in C$ such that $d(u,v) = i$. The *distance distribution* of C is the $(n+1)$-tuple of rational numbers $(A_0,A_1,\dots,A_n)$ defined by

$$A_i = \frac{1}{M} \sum_{u \in C} A_i(u), \qquad i = 0,\dots,n.$$

Then $A_0 = 1$, $A_i \geq 0$ and $\sum_i A_i = M \leq A(n,d)$.

## 2. BOUNDS ON A(n,d)

The first theorem is immediate, while the second gives $A(n,d)$ exactly if $n \leq 2d$.

THEOREM 1.

$$A(n-1,2\delta-1) = A(n,2\delta),$$
$$A(n,d) \leq 2A(n-1,d).$$

THEOREM 2. (PLOTKIN [48] & LEVENSHTEIN [39]). *Provided certain Hadamard matrices of order* $\le n$ *exist,**)

$$A(n,2\delta) = 2\left[\frac{2\delta}{4\delta-n}\right] \qquad if\ 4\delta > n \ge 2\delta, \tag{1}$$

$$A(4\delta,2\delta) = 8\delta, \qquad A(n,2\delta) = 1 \qquad if\ n < 2\delta. \tag{2}$$

The linear programming approach is based on the following theorem.

THEOREM 3. (DELSARTE [8]-[10]). *Let* C *be an* (n,M,d) *code with distance distribution* $(A_0,\ldots,A_n)$. *Then the quantities* $B_0,\ldots,B_n$ *are nonnegative, where*

$$B_k = M^{-1} \sum_{i=0}^{n} A_i K_k(i) \qquad (k = 0,1,\ldots,n),$$

*and* $K_k$ *is a Krawtchouk polynomial, defined by*

$$K_k(t) = \sum_{j=0}^{k} (-1)^j \binom{n-t}{k-j}\binom{t}{j} \qquad (k = 0,1,\ldots,n).$$

For later reference we give a short proof.

PROOF. Let w be a word in $\{0,1\}^n$ of weight i. Then it is easily checked that

$$\sum_{\substack{x\in\{0,1\}^n\\ wt(x)=k}} (-1)^{\langle w,x\rangle} = K_k(i).$$

Consequently by the definition of $A_i$,

$$B_k = M^{-1} \sum_{i=0}^{n} A_i K_k(i) = M^{-2} \sum_{i=0}^{n} \sum_{\substack{u,v\in C\\ wt(u-v)=i}} \sum_{\substack{x\in\{0,1\}^n\\ wt(x)=k}} (-1)^{\langle u-v,x\rangle} =$$

$$= M^{-2} \sum_{\substack{x\in\{0,1\}^n\\ wt(x)=k}} b_x^2 \ge 0, \tag{3}$$

*) Hadamard matrices are known to exist for all orders ≤ 264. In any case the r.h.s. is an upper bound on both (1) and (2).

where

$$b_x = \sum_{u \in C} (-1)^{<u,x>}. \tag{3a}$$

NOTE. If $C$ is a linear code, then $b_x$ equals M or 0 depending on whether x belongs to the dual code or not, and $B_0,\ldots,B_n$ is the weight distribution of the dual code.

To apply Theorem 3 let $C$ be an optimal code of length n and minimum distance d. Then

$$M = A(n,d) = 1 + A_d + A_{d+1} + \ldots + A_n.$$

Suppose $L^*(n,d)$ is the optimal solution to the following linear programming problem.

*Choose real variables* $A_d, A_{d+1},\ldots,A_n$ *so as to maximize* $L = A_d + A_{d+1} + \ldots + A_n$ *subject to the constraints*

$$A_i \geq 0, \qquad i = d,\ldots,n,$$
$$B_k \geq 0, \qquad k = 0,\ldots,n,$$

where

$$B_k = M^{-1}(K_k(0) + \sum_{i=d}^{n} A_i K_k(i)).$$

Then plainly

$$A(n,d) \leq 1 + L^*(n,d).$$

This is the simplest version of the *linear programming bound* for binary codes (DELSARTE [8]).

Often it is possible to impose additional constraints on the $B_i$. Certainly

$$B_i \leq A(n,d,i), \tag{4}$$

so bounds on $A(n,d,w)$ can be used (see Table 2). Sometimes several such bounds can be combined, as the following example illustrates.

THEOREM 4. $A(13,6) = 32$, *and so the Nadler code is optimal.*

PROOF. In 1959 R.F. STEVENS & W.G. BOURICIUS [60] found (13,32,6) and (14,64,6) codes, showing that $A(13,6) \geq 32$. The former code was rediscovered by NADLER [45], and is usually referred to as the Nadler code. (See also VAN LINT [41].)

To prove $A(13,6) \leq 32$ we proceed as follows.

First observe that if we shorten a (13,M,6) code and then add an overall parity check, we get a (13,M,6) code $C$ in which all distances are even.

If $(A_i)$ is the distance distribution of $C$ then $A_0 = 1$ and the remaining $A_i$'s are zero except (possibly) for $A_6$, $A_8$, $A_{10}$ and $A_{12}$. The inequalities $B_k \geq 0$ become

$$\begin{aligned}
13 + A_6 - 3A_8 - 7A_{10} - 11A_{12} &\geq 0.\\
\binom{13}{2} - 6A_6 - 2A_8 + 18A_{10} + 54A_{12} &\geq 0.\\
\binom{13}{3} - 6A_6 + 14A_8 - 14A_{10} - 154A_{12} &\geq 0.\\
\binom{13}{4} + 15A_6 - 5A_8 - 25A_{10} + 275A_{12} &\geq 0.\\
\binom{13}{5} + 15A_6 - 25A_8 + 63A_{10} - 197A_{12} &\geq 0.\\
\binom{13}{6} - 20A_6 + 20A_8 - 36A_{10} + 132A_{12} &\geq 0.
\end{aligned} \tag{5}$$

Furthermore we have

$$A_{12}(u) \leq A(13,6,12) = A(13,6,1) = 1,$$
$$A_{10}(u) \leq A(13,6,10) = A(13,6,3) = 4.$$

However, these can be combined. For if $A_{12}(u) = 1$ then $A_{10}(u) = 0$. So

$$A_{10}(u) + 4A_{12}(u) \leq 4,$$

and averaging over u gives

$$A_{10} + 4A_{12} \leq 4. \tag{6}$$

Actually (6) and the first two constraints of (5) turn out to be enough, and so we consider the problem:

$$\text{Maximize } A_6 + A_8 + A_{10} + A_{12}$$

subject to

$$A_6 \geq 0, A_8 \geq 0, A_{10} \geq 0, A_{12} \geq 0$$

and

$$\begin{aligned} 13 + A_6 - 3A_8 - 7A_{10} - 11A_{12} &\geq 0. \\ 78 - 6A_6 - 2A_8 + 18A_{10} + 54A_{12} &\geq 0, \\ 4 - A_{10} - 4A_{12} &\geq 0. \end{aligned} \tag{7}$$

The dual problem is:

$$\text{Minimize } 13u_1 + 78u_2 + 4u_3$$

subject to

$$u_1 \geq 0, u_2 \geq 0, u_3 \geq 0$$

and

$$\begin{aligned} 1 + u_1 - 6u_2 &\leq 0, \\ 1 - 3u_1 - 2u_2 &\leq 0, \\ 1 - 7u_1 + 18u_2 - u_3 &\leq 0, \\ 1 - 11u_1 + 54u_2 - 4u_3 &\leq 0. \end{aligned} \tag{8}$$

Feasible solutions to these two problems are

$$A_6 = 24, A_8 = 3, A_{10} = 4, A_{12} = 0, \tag{9}$$

$$u_1 = u_2 = \frac{1}{5}, u_3 = \frac{16}{5}. \tag{10}$$

In fact since the corresponding objective functions are equal:

$$24 + 3 + 4 + 0 = 13 \cdot \frac{1}{5} + 78 \cdot \frac{1}{5} + 4 \cdot \frac{16}{5} = 31,$$

it follows that (9) and (10) are optimal solutions. (These solutions are easily obtained by hand using the simplex method - see [18] or [52].) It follows that $A(12,5) = A(13,6) \le 32$. □

REMARK. The following argument shows that (9) is the unique optimal solution. Let $x_6, x_8, x_{10}, x_{12}$ be any optimal solution to the primal problem. The $u_i$ of (10) are all positive and satisfy the first three constraints of (8) with equality but not the fourth. Hence from the theorem of complementary slackness (SIMONNARD [52]) the $x_i$ must satisfy the primal constraints (7) with equality, and $x_{12} = 0$. These three equations have the unique solution

$$x_6 = 24,\ x_8 = 3,\ x_{10} = 4.$$

Thus (9) is the unique optimal solution. Therefore the distance distribution of a (13,32,6) code in which all distances are even is unique. This result has been used by GOETHALS [19] to show that the code itself is unique and that there are exactly two nonequivalent (12,32,5) codes. (cf. NADLER [45], VAN LINT [41]).

If $A(n,d) \not\equiv 0 \pmod 4$, the right hand side of the Delsarte inequalities $B_k \ge 0$ can sometimes be increased, as shown by Theorems 5 and 8.

THEOREM 5. *Let* C *be an* (n,M,d) *code with* $M = A(n,d)$, *and suppose that* M *is odd. Then*

$$B_k \ge M^{-2}\binom{n}{k} \qquad (k = 0,1,\ldots,n).$$

PROOF. If M is odd, then $b_x$ (Eq. (3a)) is odd, and hence non-zero. From (3) we get

$$B_k \ge M^{-2} \sum_{\substack{x\in\{0,1\}^n \\ wt(x)=k}} b_x^2 \ge M^{-2}\binom{n}{k}.$$

REMARK. The first term in $B_k = M^{-1} \sum_{i=0}^{n} A_i K_k(i)$ is $M^{-1} K_k(0) = M^{-1}\binom{n}{k}$. Hence Theorem 5 shows that, in all the Delsarte inequalities, the constant term may be multiplied by $(M-1)/M$. That means that - if no extra inequalities have been added - the optimal solution is simply $(M-1)/M$ times the original one, and hence $\sum_{i=1}^{n} A_i < M-1$, lowering the bound by exactly one. If extra inequalities are added, the gain is in general less.

As an application we prove:

THEOREM 6. $A(9,4) = 20$.

PROOF. GOLAY [21] found a (9,20,4) code, thus $A(9,4) \geq 20$. A cyclic (8,20,3) code is given in SLOANE & WHITEHEAD [57]. To prove $A(9,4) \leq 20$, as usual let $C^-$ be an $(8,M,3)$ code with $M = A(8,3) = A(9,4)$; and let $C$ be the $(9,M,4)$ extended code, having distance distribution $(A_0,\ldots,A_9)$ with $A_0 = 1$ and $A_1 = A_2 = A_3 = A_5 = A_7 = A_9 = 0$.

First we maximize $A_4 + A_6 + A_8$ subject to $A_i \geq 0$, $B_k \geq 0$ and $A_8 \leq 1$, obtaining $A_4 + A_6 + A_8 \leq 20\frac{1}{3}$, hence $M \leq 21$.

Suppose $M = 21$. Then by Theorem 5 we can replace $B_k \geq 0$ by $B_k \geq \frac{1}{441}\binom{9}{k}$. Moreover, it is obvious that $A_8 \leq \frac{20}{21}$. Hence in this case, in spite of the extra inequality, all constant terms occurring in the inequalities are multiplied by $\frac{20}{21}$, so

$$M \leq 1 + \frac{20}{21} * 20\frac{1}{3} < 21.$$

Hence $A(9,4) = 20$. □

If $A(n,d) \equiv 2 \pmod 4$, then a positive lower bound for $B_k$ can be obtained by noting that $b_x$ cannot be zero too often. For example if $u_1$, $u_2$, $u_1+u_2$ are distinct then $b_{u_1}$, $b_{u_2}$, $b_{u_1+u_2}$ cannot all be zero. The following linear inequality can be obtained in this way.

THEOREM 7. *Let* $C$ *be an* $(n,M,d)$ *code with* $M = A(n,d)$, *and suppose that* $M \equiv 2 \pmod 4$. *Then*

$$B_k \geq \frac{4}{3M^2}\binom{n}{k},$$

*If* (i) $k$ *is even and* $0 \le k \le \frac{2}{3}n$, *or* (ii) *if* $d$ *is even,* $k \equiv n \pmod 2$, *and* $\frac{1}{3}n \le k \le n$.

A slightly stronger result is:

THEOREM 8. *Let* $C$ *be an* $(n,M,d)$ *code with* $M = A(n,d)$, *and suppose that* $M \equiv 2 \pmod 4$. *Then there exists an* $\ell \in \{0,1,\dots,n\}$ *such that*

$$B_k \ge 2M^{-2}\left(\binom{n}{k} + K_k(\ell)\right). \qquad (k = 0,1,\dots,n).$$

(Since $|K_k(\ell)| \le \binom{n}{k}$, this also improves Theorem 3.)

PROOF. Since $M$ is even, $b_u$ is even for each $u \in \{0,1\}^n$. Let $e_j$ be the j-th unit vector in $\{0,1\}^n$. Then

$$b_x - b_{x+e_j} = \sum_{u \in C} (1-(-1)^{\langle u,e_j\rangle})(-1)^{\langle u,x\rangle}.$$

Hence, for fixed $j$, the residue class of $b_x - b_{x+e_j} \pmod 4$ is even and independent of the choice of $x$.

Let $J$ be the set of those $j \in \{1,2,\dots,n\}$ for which $b_x - b_{x+e_j} \equiv 2 \pmod 4$, and let $\ell = |J|$ and $\xi = \sum_{j \in J} e_j$. Then

$$b_x - b_{x+e_j} \equiv 2\langle e_j,\xi\rangle \pmod 4$$

By induction on the weight of $x$ it follows that, since $b_0 = M \equiv 2 \pmod 4$,

$$b_x \equiv 2 + 2\langle x,\xi\rangle \pmod 4$$

Now for each $k \in \{0,1,\dots,n\}$

$$B_k = M^{-2} \sum_{\substack{x \in \{0,1\}^n \\ wt(x)=k}} b_x^2 \ge 2M^{-2} \sum_{\substack{x \in \{0,1\}^n \\ wt(x)=k}} (1+(-1)^{\langle x,\xi\rangle}) =$$

$$= 2M^{-2}\left(\binom{n}{k} + K_k(\ell)\right). \qquad \square$$

We mention the following immediate consequence of Theorem 8, which is weaker but easier to apply.

COROLLARY. *Let* C *be an* (n,M,d) *code with* M = A(n,d), *and suppose that* $M \equiv 2 \pmod 4$. *Then*

$$B_k \geq 2M^{-2}\left(\binom{n}{k} + \min_{\ell \in \{0,1,\dots,n\}} K_k(\ell)\right).$$

E.g. $B_2 \geq \frac{4}{M^2}\left(\binom{n}{2} - \left[\frac{n^2}{4}\right]\right)$.

For example, this corollary can be used to prove the upper bound in Theorem 9; the lower bound comes from [56], [57].

THEOREM 9. A(17,8) = 36 *or* 37. □

Table 1 gives the bounds on A(n,d). Many values come from Theorems 1 and 2. Otherwise the unmarked upper bounds are obtained by linear programming, as illustrated in Theorems 4 and 6. Other entries are explained by the key. The bounds $A(9,4) \leq 20$, $A(10,4) \leq 39$, $A(11,4) \leq 78$, and $A(12,4) \leq 154$ were claimed by WAX [63] in 1959. However, as we shall see in the next section, such bounds cannot be obtained by his method.

We conclude this section by repeating Elspas's question [16]: can A(n,d) be odd and greater than 1? From Theorem 2 and Table 1 we have:

THEOREM 10. *If* A(n,d) *is odd (and* > 1) *then* $A(n,d) \geq 37$. *If Hadamard matrices exist of all orders, then* A(n,d) *is even whenever* $n \leq 2d$. □

## 3. THE END OF THE WAX BOUND

In 1959 N.WAX [63] computed a number of upper bounds for binary codes by a method derived from sphere packing in Euclidean spaces as developed by R.A. RANKIN [49] (see also ROGERS [50]). Most of the bounds obtained were rather weak, but there were three special cases in which his "soft sphere model" seemingly yielded astonishingly good results. These were:

$$A(8,3) \le 20,$$
$$A(9,3) \le 39 \text{ (and hence } A(10,3) \le 78),$$
$$A(11,3) \le 154.$$

The first bound is confirmed by Theorem 6, but no proof of the other bounds is known.

We were unable to duplicate Wax's calculations, and in fact in this section we shall establish a lower bound on the best upper bound that can be achieved with the soft sphere model, no matter which weight function is used. Since this lower bound is inconsistent with the data found by Wax, we may conclude that Wax's results are - at least in the interesting cases mentioned above - erroneous.

We are now left with the following bounds for $A(8,3)$, $A(9,3)$, $A(10,3)$ and $A(11,3)$:

$$A(8,3) = 20$$
$$38 \le A(9,3) \le 40$$
$$72 \le A(10,3) \le 80$$
$$144 \le A(11,3) \le 160$$

## 3.1. THE SOFT SPHERE MODEL

Consider an $(n,M,d)$ code as a subset of the vertices of the hypercube $[0,1]^n$ in Euclidean n-space $\mathbb{R}^n$. The Euclidean distance between two codepoints is at least $\sqrt{d}$. Therefore the spheres with centers in the codepoints and radii $R = \frac{1}{2}\sqrt{d}$ are disjoint. If $V$ denotes the volume of the intersection of each sphere with the hypercube $[0,1]^n$ (by symmetry these volumes are all equal), then the number of codepoints evidently cannot exceed $1/V$. Hence $A(n,d) \le [1/V]$.

This method, called the "hard sphere model", yields very modest results, e.g. $A(9,3) \le 566$ (and not 56.7 as in WAX [63]) or $A(10,4) \le 401$.

In order to sharpen the bounds, the hard spheres are replaced by larger ones with variable mass density. As basic conditions it is required that

(i) the density $\rho(r)$ associated with a single sphere is non-negative and depends only on the distance $r$ to the center of that sphere and that

(ii) in any configuration of (partly overlapping) spheres with centers at least 2R apart, the total density at each point does not exceed unity.

If $\mu$ is the mass of the intersection of each sphere with the hypercube [1], we now obtain:

$$A(n,d) \leq [1/\mu].$$

The main problem is to determine a suitable density which satisfies the basic conditions (i) and (ii), and optimizes the mass $\mu$. R.A. RANKIN studied this problem in [49]. In order to simplify computations, he required in addition:

(iii) The spheres have radius $R\sqrt{2}$, i.e. $\rho(r) = 0$ if $r \geq R\sqrt{2}$.

The model described, with the conditions (i), (ii) and (iii), is called the "soft sphere model". We shall denote the least upper bound for A(n,d) that can be achieved with this model by $A_w(n,d)$. Our aim is to give a lower bound for $A_w(n,d)$.

## 3.2. A LOWER BOUND FOR $A_w(n,d)$

First we derive an upper bound for $\rho$. We define for each positive integer m:

$$y_m = \sqrt{2(m-1)/m}$$

(note: $y_1 = 0$, $y_2 = 1$), and the function $\sigma$: $[0,\infty] \to [0,1]$ by

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1) In case $d \leq 4$ one may instead define $\mu$ by $2^{-n}$ times the mass of the whole sphere, since the configuration may be continued with period 2 in all directions in $\mathbb{R}^n$. But even this extended model is included in our results, since we estimate $\mu$ by that number.

$$\sigma(r) = \frac{1}{m} \quad \text{if } Ry_m \le r < Ry_{m+1} \qquad (m = 1,2,\dots,n),$$
$$= \frac{1}{n+1} \quad \text{if } Ry_{n+1} \le r < R\sqrt{2},$$
$$= 0 \quad \text{if } r \ge R\sqrt{2}.$$

Then we have:

LEMMA 11. $\rho \le \sigma$.

PROOF. We have to prove that $\rho(r) \le 1/m$ if $r \ge Ry_m$ for $m = 1,2,\dots,n+1$. Let $m \in \{1,2,\dots,n+1\}$. Suppose m spheres with density function $\rho$ are arranged such that their centers form the vertices of an (m-1)-dimensional regular simplex in $\mathbb{R}^n$ with edges of length 2R. Then the distance from the center of gravity of the simplex to each of the vertices equals

$$R\sqrt{2(m-1)/m} = Ry_m.$$

(Proof by induction.)

The total density at the center of gravity equals $m\rho(Ry_m)$. Hence $\rho(Ry_m) \le 1/m$ and a fortiori $\rho(r) \le 1/m$ if $r \ge Ry_m$. □

This estimate for $\rho$ immediately gives rise to an upper bound on the mass $\mu$:

LEMMA 12. $\mu \le \left(\frac{\pi e R^2}{n}\right)^{\frac{1}{2}n} \frac{1}{\sqrt{\pi n}} \left( \sum_{m=1}^{n} \frac{1}{m(m+1)} \left(\frac{m}{m+1}\right)^{\frac{1}{2}n} + \frac{1}{n+1}\right).$

PROOF. We denote the volume of the intersection of the n-dimensional hypersphere with radius r and center 0 in $\mathbb{R}^n$ and the n-dimensional hypercube $[0,1]^n$ by B(r). The volume of the n-dimensional unit sphere will be denoted by $J_n$. It is well known that

$$J_n = \frac{\pi^{\frac{1}{2}n}}{(\frac{1}{2}n)!} \le \frac{\pi^{\frac{1}{2}n} e^{\frac{1}{2}n}}{(\frac{1}{2}n)^{\frac{1}{2}n}\sqrt{\pi n}} = \left(\frac{2\pi e}{n}\right)^{\frac{1}{2}n} \frac{1}{\sqrt{\pi n}}.$$

Hence

$$\mu = \int_0^{R\sqrt{2}} \rho(r)\, dB(r) \le \int_0^{R\sqrt{2}} \sigma(r)\, dB(r) =$$

$$= -\int_0^{R\sqrt{2}} B(r)\, d\sigma(r) \le -\int_0^{R\sqrt{2}} 2^{-n} J_n\, r^n\, d\sigma(r) =$$

$$= 2^{-n} J_n \left( \sum_{m=2}^{n+1} \left( \frac{1}{m-1} - \frac{1}{m} \right) (Ry_m)^n + \frac{1}{n+1} (R\sqrt{2})^n \right) \le$$

$$\le \left(\frac{R}{2}\right)^n \left(\frac{2\pi e}{n}\right)^{\frac{1}{2}n} \frac{1}{\sqrt{\pi n}} \left( \sum_{m=2}^{n+1} \frac{1}{(m-1)m} \left(\frac{2(m-1)}{m}\right)^{\frac{1}{2}n} + \frac{2^{\frac{1}{2}n}}{n+1} \right) =$$

$$= \left(\frac{\pi e R^2}{n}\right)^{\frac{1}{2}n} \frac{1}{\sqrt{\pi n}} \left( \sum_{m=1}^{n} \frac{1}{m(m+1)} \left(\frac{m}{m+1}\right)^{\frac{1}{2}n} + \frac{1}{n+1} \right). \quad \square$$

This leads to the lower bound for $A_w(n,d)$:

THEOREM 13. $A_w(n,d) \ge \left[ \left( \left(\frac{4n}{\pi e d}\right)^{\frac{1}{2}n} \sqrt{\pi n} \left( \sum_{m=1}^{n} \frac{1}{m(m+1)} \left(\frac{m}{m+1}\right)^{\frac{1}{2}n} + \frac{1}{n+1} \right) \right)^{-1} \right]$.

PROOF. $R = \frac{1}{2}\sqrt{d}$ and $A_w(n,d) = [1/\mu]$ for some density function $\rho$. $\square$

EXAMPLES. $A_w(8,3) \ge 45$, $A_w(9,4) \ge 27$,
$A_w(9,3) \ge 101$, $A_w(10,4) \ge 56$,
$A_w(10,3) \ge 238$, $A_w(11,4) \ge 119$,
$A_w(11,3) \ge 579$, $A_w(12,4) \ge 259$.

## 4. BOUNDS ON A(n,d,w)

The first two theorems are well-known.

THEOREM 14. *Let* d,w,n *be integers,* $d \ne 0$, $w \le n$. *Then*

(i) $A(n,d-1,w) = A(n,d,w)$ *if* d *is even,*

(ii) $A(n,d,w) = A(n,d,n-w)$,

(iii) $A(n,d,w) = 1$ *if* $d > 2w$,

(iv) $A(n,d,w) = \left[\frac{n}{w}\right]$ *if* $d = 2w$.

THEOREM 15. *If a* 2d × 2d *Hadamard matrix exists,*

$$A(2d-2,d,d-1) = d,$$
$$A(2d-1,d,d-1) = 2d-1,$$
$$A(2d,d,d) = 4d-2.$$

Theorems 16-18 are due to JOHNSON [23], [24].

THEOREM 16.

$$A(n,d,w) \le \left[\frac{dn}{dn-2w(n-w)}\right]$$

*provided the denominator is positive.*

A slightly stronger result is:

THEOREM 17. *Suppose* $A(n,d,w) = M$, *and define* $q$ *and* $r$ *by*

$$wM = nq + r, \quad 0 \le r < n.$$

*Then*

$$nq(q-1) + 2qr \le (w-\tfrac{1}{2}d)M(M-1),$$

THEOREM 18.

$$A(n,d,w) \le \left[\frac{n}{w} A(n-1,d,w-1)\right] \qquad (n \ge w \ge 1)$$

$$A(n,d,w) \le \left[\frac{n}{n-w} A(n-1,d,w)\right] \qquad (n > w \ge 0)$$

THEOREM 19.

$$A(n,d,w) \le \frac{n}{w} \cdot \frac{n-1}{w-1} \cdot \ldots \cdot \frac{n-t+1}{w-t+1} \cdot A(n-t,d,w-t) \qquad (n \ge w \ge t).$$

*If equality holds then any optimal constant weight code with parameters* $n,d,w$ *is a* $t$-*design. In particular*

$$A(n,2\delta,w) = \frac{n(n-1)\ldots(n-w+\delta)}{w(w-1)\ldots\delta}$$

*if and only if a Steiner system* S(w-δ+1,w,n) *exists.*
(For a bibliography of Steiner systems up to 1973 see DOYEN and ROSA [14]).

## 4.1. OPTIMAL CONSTANT WEIGHT CODES

As noted in the introduction the determination of A(n,d,w) is equivalent to determining D(t,d,v), where $v = n$, $k = w$ and $t = k + 1 - \frac{1}{2}d$ (if d is even). But this requires the construction of (maximal partial) Steiner t-designs, which is trivial for $t = 1$, while for $t = 2$ the recursive techniques of HANANI and WILSON are available (see e.g. WILSON [64,65]). For larger t almost nothing is known (the best studied case being $t = 3$, $k = 4$). The known results are as follows:

1. $t = 1$.
   This is Theorem 14 (iv): $A(n,2w,w) = \left[\frac{n}{w}\right]$.
2. $t = 2$.
   In this case we must look for a maximal collection of w-subsets of an n-set such that no 2-subset is covered twice (in other words: an edge-disjoint packing of w-cliques in the complete graph on n points). If a BIBD $(b,v{=}n,r,k{=}w,\lambda{=}1)$ exists (that is, an S(2,w,n)) then obviously $A(n,d,w) = b = \binom{n}{2}/\binom{w}{2}$; otherwise we must look for the nearest approximation to this Steiner system.

2.1 $d = 4$, $w = 3$.
It has been shown by KIRKMAN [38] in the cases $n \equiv 0, 1, 2$ or $3 \pmod 6$ and by SCHÖNHEIM [51] in the remaining cases that

$$A(n,4,3) = \begin{cases} \left[\frac{n}{3}\left[\frac{n-1}{2}\right]\right] & \text{for } n \not\equiv 5 \pmod 6 \\ \left[\frac{n}{3}\left[\frac{n-1}{2}\right]\right] - 1 & \text{for } n \equiv 5 \pmod 6 \end{cases}$$

(see also GUY [22], SPENCER [58] and SWIFT [61]). The cases $n \equiv 1$ or $3 \pmod 6$ correspond to Steiner triple systems.

2.2 $d = 6$, $w = 4$.
As has been shown by HANANI [26] there exist Steiner systems S(2,4,n) iff $n \equiv 1$ or $4 \pmod{12}$. In BROUWER & SCHRIJVER [7] group divisible designs GD(4,1,2;n) are constructed for each $n \equiv 2 \pmod 6$, $n \neq 8$. In BROUWER

[5] pairwise balance designs PBD($\{4,7^*\}$;n) are constructed for each $n \equiv 7$ or $10 \pmod{12}$, $n \neq 10,19$. Using these and similar constructions it follows that if

$$JB(n,6,4) := \begin{cases} \left[\frac{n}{4}\left[\frac{n-1}{3}\right]\right] - 1 & \text{for } n \equiv 7 \text{ or } 10 \pmod{12} \\ \left[\frac{n}{4}\left[\frac{n-1}{3}\right]\right] & \text{otherwise} \end{cases}$$

then $A(n,6,4) = JB(n,6,4)$ for all n with the exception of $n = 8,9,10,11,17,19$. The values of $A(n,6,4)$ for $n = 8,9,10,11$ are easily determined by hand, that of $A(17,6,4)$ was determined in BROUWER [4], and the lower bound on $A(19,6,4)$ follows from a construction of H.R. PHINNEY [47].

We conjecture that for $t = 2$, w fixed and n sufficiently large (i.e. $n \geq n_0(w)$) $A(n,d,w)$ equals the Johnson bound (obtained by applying Theorems 14 and 18) (cf. WILSON [64]).

2.3 $d = 8$, $w = 5$.

As shown by HANANI [26,27] there exist Steiner systems $S(2,5,n)$ iff $n \equiv 1$ or $5 \pmod{20}$. Shortening these gives optimal codes for $n \equiv 0$ or $4 \pmod{20}$.

$A(n,8,5)$ for $n \leq 15$ follows from the following observation:

<u>THEOREM 20</u>. *If* d *is even,* $\lambda = w-\frac{1}{2}d$, $M \leq \frac{w}{\lambda} + 1$ *then* $A(n,d,w) \geq M$ iff $n \geq wM - \lambda\binom{w}{2}$.

Many more values of $A(n,8,5)$ are known, but most lie outside the range of the table.

3.1 $t = 3$, $d = 4$, $w = 4$.

As is shown by HANANI [25] Steiner quadruple systems exist for each $n \equiv 2$ or $4 \pmod 6$. Hence for these values of n we have $A(n,4,4) = \frac{1}{4}\binom{n}{3}$.

Shortening these codes once gives $A(n,4,4) = \frac{1}{24}n(n-1)(n-3)$ for $n \equiv 1$ or $3 \pmod 6$. Using triplewise balanced designs TBD($\{4,6\}$;n) in which the blocks of size 6 form a partition it follows that $A(n,4,4) = \frac{1}{24}n(n^2-3n-6)$ for $n \equiv 0 \pmod 6$ (cf. BROUWER [6]). Exact values for the case $n \equiv 5 \pmod 6$ are not yet known.

## 4.2. THE LINEAR PROGRAMMING BOUND FOR A(n,d,w)

This is based on:

THEOREM 21. (DELSARTE [9],[10]). *Let C be an* $(n,M,2\delta)$ *code of constant weight* $w \le \frac{1}{2}n$, *having distance distribution* $(A_0,\ldots,A_{2w})$. *Then the quantities* $B_0,\ldots,B_{2w}$ *are nonnegative, where now*

$$B_{2k} = \frac{1}{M}\sum_{i=0}^{w} A_{2i}Q_k(i,n,w) \qquad k = 0,\ldots,w,$$

*the coefficients* $Q_k(i,n,w)$ *are given by*

$$Q_k(i,n,w) = \frac{n-2k+1}{n-k+1} E_i(k)\binom{n}{k}\Big/\binom{w}{i}\binom{n-w}{i}, \tag{11}$$

*and* $E_i(x)$ *is an Eberlein (or dual Hahn) polynomial defined by*

$$E_i(x) = \sum_{j=0}^{i} (-1)^{i-j}\binom{w-j}{i-j}\binom{w-x}{j}\binom{n-w+j-x}{j}.$$

(See DELSARTE [9], EBERLEIN [15], HAHN [23] and KARLIN & McGREGOR [37] for these polynomials.)

As in the case of A(n,d) we obtain a bound on A(n,d,w) by maximizing $A_0+A_2+\ldots+A_{2w}$ subject to the constraints

$$A_{2i} \ge 0 \qquad (i = \delta,\ldots,w),\ A_0 = 1,\ A_2 = A_4 = \ldots = A_{2\delta-2} = 0, \tag{12}$$

and

$$B_{2k} \ge 0 \qquad (k = 0,\ldots,w). \tag{13}$$

Additional constraints on the $A_i$ can be expressed in terms of the function $T(w_1,n_1,w_2,n_2,2\delta)$ defined in section 1 (see Table 3). Let $u \in C$ and consider the codewords $v \in C$ such that dist(u,v) = 2i. By a suitable permutation of the coordinates we may assume that

```
        <------ w ------> <------n-w------>
    u = 11···1  11···1  00···0  00···0
    v = 11···1  00···0  11···1  00···0
                <--i-->  <--i-->
```

The number of such v's is $A_{2i}(u)$, and by definition of T we have

$$A_{2i}(u) \le T(i,w,i,n-w,2\delta), \quad i = \delta,\dots,w,$$

so that

$$A_{2i} \le T(i,w,i,n-w,2\delta), \quad i = \delta,\dots,w. \tag{14}$$

Sometimes it is possible to say more, as the following example illustrates.

THEOREM 22.

$$A(17,8,7) \le 31.$$

PROOF. Let $C$ be a code of length 17, distance 8 and constant weight 7. Suppose $C$ contains $M = A(17,8,7)$ codewords. For $u \in C$, the only nonzero components of the weight distribution with respect to u are $A_0(u) = 1$, $A_8(u)$, $A_{10}(u)$, $A_{12}(u)$, $A_{14}(u)$, and then

$$A_i = \frac{1}{M} \sum_{u \in C} A_i(u), \quad i = 0,8,10,12,14.$$

We have

$$A_{14}(u) \le A(10,8,7) = A(10,8,1) = 1,$$
$$A_{12}(u) \le T(6,7,6,10,8) = T(1,7,4,10,8) = 5.$$

These imply $A_{12} \le 5$, $A_{14} \le 1$ as in (14). But we can say more. For if $A_{14}(u) = 1$ then $A_{12}(u) \le 2$. Therefore, for all $u \in C$,

$$A_{12}(u) + 3A_{14}(u) \le 5 \text{ and } A_{14}(u) \le 1,$$

and so

$$A_{12} + 3A_{14} \le 5 \text{ and } A_{14} \le 1. \tag{15}$$

Linear programming with the constraints (12), (13), (15) gives the stated result. □

Table 2 gives the bounds on $A(n,d,w)$. Upper bounds marked with an L are obtained by linear programming, as illustrated by Theorem 22. Unmarked lower and upper bounds are from Theorems 14-20. A useful technique for getting lower bounds is the following. Let $C$ be an $(n,M,d)$ code, and $C^* = a + C = \{a+u: u \in C\}$ any translate of $C$, with weight distribution $A_i(\underline{0})$. Then

$$A_i(\underline{0}) \le A(n,d,i).$$

This technique works well for example with the (shortened) Nordstrom-Robinson and Golay codes. Other entries in the table are explained by the key. Letters on the left of an entry refer to lower bounds, on the right to upper bounds.

## 5. BOUNDS ON $T(w_1,n_1,w_2,n_2,d)$

$T(w_1,n_1,w_2,n_2,d)$ is the maximum number of binary vectors of length $n_1 + n_2$, having mutual Hamming distance at least d, where each vector has exactly $w_1$ ones in the first $n_1$ coordinates and exactly $w_2$ ones in the last $n_2$ coordinates. For example, $T(1,3,2,4,6) = 2$, as illustrated by the vectors 1001100, 0100011. Properties of this function are given in the following theorems.

<u>THEOREM 23</u>. (JOHNSON [34]).

(a) $T(w_1,n_1,w_2,n_2,d) = T(w_2,n_2,w_1,n_1,d)$.

(b) $T(w_1,n_1,w_2,n_2,d) = T(n_1-w_1,n_1,w_2,n_2,d)$,

(c) $T(0,n_1,w_2,n_2,d) = A(n_2,d,w_2)$,

(d) $T(w_1,n_1,w_2,n_2,d) \le A(n_2,d-2w_1,w_2)$,

(e) *If* $d = 2w_1 + 2w_2$ *then*

$$T(w_1,n_1,w_2,n_2,d) = \min\left\{\left[\frac{n_1}{w_1}\right],\left[\frac{n_2}{w_2}\right]\right\},$$

(f) $T(w_1,n_1,w_2,n_2,d) \le \left[\frac{n_1}{w_1} T(w_1-1,n_1-1,w_2,n_2,d)\right]$,

(g) $T(w_1,n_1,w_2,n_2,d) \le \left[\frac{n_1}{n_1-w_1} T(w_1,n_1-1,w_2,n_2,d)\right]$,

(h) $T(w_1,n_1,w_2,n_2,2\delta) \le \left[\dfrac{\delta}{\dfrac{w_1^2}{n_1}+\dfrac{w_2^2}{n_2}+\delta-w_1-w_2}\right]$,

*provided the denominator is positive.*

A slightly stronger result than Theorem 23 (h) is:

THEOREM 24. *Suppose* $T(w_1,n_1,w_2,n_2,2\delta) = M$, *and define* $q_i, r_i$ $(i = 1,2)$ *by*

$$Mw_i = q_i n_i + r_i, \qquad 0 \le r_i < n_i.$$

*Then*

$$\sum_{i=1}^{2} \{n_i q_i (q_i-1) + 2q_i r_i\} \le (w_1+w_2-\delta)M(M-1),$$

*with equality if and only if all distances are* $2\delta$.

There is also a linear programming bound for $T(w_1,n_1,w_2,n_2,2\delta)$, based on Theorem 25. Define the *left* and *right weights* of a vector $\underline{u} = (u_1,\ldots,u_{n_1+n_2})$ to be $w_L(\underline{u}) = wt(u_1,\ldots,u_{n_1})$ and $w_R(\underline{u}) =$ $= wt(u_{n_1+1},\ldots,u_{n_2})$.

THEOREM 25. *Let C be an* $(n_1+n_2,M,2\delta)$ *code such that* $w_L(\underline{u}) = w_1$, $w_R(\underline{u}) = w_2$ *for all* $u \in C$, *and let*

$$A_{2i,2j}(u) = |\{v \in C\colon w_L(u+v) = 2i,\ w_R(u+v) = 2j\}|,$$
$$A_{2i,2j} = \frac{1}{M}\sum_{u\in C} A_{2i,2j}(u).$$

*Then*

$$B_{2k,2\ell} = \frac{1}{M}\sum_{i=0}^{w_1}\sum_{j=0}^{w_2} A_{2i,2j}\, Q_k(i,n_1,w_1)Q_\ell(j,n_2,w_2) \ge 0,$$

*where* $Q_k(i,n,w)$ *is given in* (11).

PROOF. For $\nu = 1,2$ suppose $(X^{(\nu)}; R_0^{(\nu)},\ldots,R_{n_\nu}^{(\nu)})$ is an association scheme with intersection numbers $p_{ijk}^{(\nu)}$, incidence matrices $D_i^{(\nu)}$, idempotents $J_i^{(\nu)}$, and eigenvalues $P_k^{(\nu)}$, $Q_k^{(\nu)}(i)$ (cf. DELSARTE [9], [10], SLOANE [54]). Then $(X^{(1)} \times X^{(2)};\ R_{ij} = R_i^{(1)} \times R_j^{(2)},\ 0 \le i \le n_1,\ 0 \le j \le n_2)$ is an association scheme (the *product scheme*) with intersection numbers $p_{ikr}^{(1)}\, p_{jls}^{(2)}$, incidence matrices $D_i^{(1)} \otimes D_j^{(2)}$, idempotents $J_i^{(1)} \otimes J_j^{(2)}$, and eigenvalues $P_k^{(1)}(i)P_\ell^{(2)}(j)$, $Q_k^{(1)}(i)Q_\ell^{(2)}(j)$. Hence $C$ is a code in the product of two Johnson schemes. The result now follows from Theorem 3.3 of DELSARTE [9] and Theorem 21 above. □

Table 3 gives upper bounds on $T(w_1,n_1,w_2,n_2,10)$. Entries marked with an asterisk (*) are exact.

## TABLE 1

Values of A(n,d)

| n | d=4 | d=6 | d=8 | d=10 |
|---|---|---|---|---|
| 6 | 4 | 2 | 1 | 1 |
| 7 | 8 | 2 | 1 | 1 |
| 8 | [a] 16 | 2 | 2 | 1 |
| 9 | [d] 20 [b] | 4 | 2 | 1 |
| 10 | [d] 38 - 40 | 6 | 2 | 2 |
| 11 | [d] 72 - 80 | 12 | 2 | 2 |
| 12 | [d] 144 - 160 | 24 | 4 | 2 |
| 13 | 256 | 32 [e] | 4 | 2 |
| 14 | 512 | 64 | 8 | 2 |
| 15 | 1024 | 128 | 16 | 4 |
| 16 | [a] 2048 | [f] 256 | 32 | 4 |
| 17 | [d] 2560 - 3276 | 256-340 | 36 - 37 [h] | 6 |
| 18 | [d] 5120 - 6552 | 512-680 | 64-74 | 10 |
| 19 | [d] 9728 - 13104 | 1024-1288 | 128-144 | 20 |
| 20 | [d] 19456 - 26208 | [g] 2048 - 2372 | 256-279 | 40 |
| 21 | [d] 36864 - 43690 | [g] 2560 - 4096 | 512 | 40-55 |
| 22 | [d] 73728 - 87380 | 4096-6942 | 1024 | [j] 48-90 |
| 23 | [d] 147456 - 173784 | 8192-13774 | 2048 | 64-150 |
| 24 | [d] 294912 - 344636 | [g] 16384 - 24106 | [i] 4096 | [k] 128 - 280 |

KEY TO TABLE 1

a. Hamming code (HAMMING [24]).

b. Theorem 6.

d. Constructed in GOLAY [21], JULIN [35] or SLOANE & WHITEHEAD [57].

e. Theorem 4.

f. Nordstrom-Robinson code, NORDSTROM & ROBINSON [46].

g. Constructed in SLOANE, REDDY & CHEN [55].

h. Theorem 9.

i. Golay code (GOLAY [20]).

j. From a (24,48,12) Hadamard code.

k. Constructed by ALLTOP [1].

# TABLE 2: A TABLE OF A(n,d,w)

## Distance 4 : A(n, 4, w)

| n \ w | 2 | 3[a] | 4[a] | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 2 | 1 | 1 | | | | | | | | |
| 5 | 2 | 2 | 1 | 1 | | | | | | | |
| 6 | 3 | 4 | 3 | 1 | 1 | | | | | | |
| 7 | 3 | 7 | 7 | 3 | 1 | 1 | | | | | |
| 8 | 4 | 8 | 14 | 8 | 4 | 1 | 1 | | | | |
| 9 | 4 | 12 | 18 | 18 | 12 | 4 | 1 | 1 | | | |
| 10 | 5 | 13 | 30 | 36 | 30 | 13 | 5 | 1 | 1 | | |
| 11 | 5 | 17 | 34-35 | 66 | 66 | 34-35 | 17 | 5 | 1 | 1 | |
| 12 | 6 | 20 | 51 | [b]73-84 | [e]132 | 73-84 | 51 | 20 | 6 | 1 | 1 |
| 13 | 6 | 26 | 65 | [b]99-132 | [d]143-182 | 143-182 | 99-132 | 65 | 26 | 6 | 1 |
| 14 | 7 | 28 | 91 | [d]143-182 | [d]210-308 | [d]216-364 | 210-308 | 143-182 | 91 | 28 | 7 |
| 15 | 7 | 35 | 105 | [d]213-272[f] | 321-455 | [d]405-660 | 405-660 | 321-455 | 213-272 | 105 | 35 |
| 16 | 8 | 37 | 140 | [d]305-336 | 513-725 | ?-1040 | ?-1320 | ?-1040 | 513-725 | 305-336 | 140 |
| 17 | 8 | 44 | 154-157 | [d]424-476 | 792-952 | ?-1760 | ?-2210 | ?-2210 | ?-1760 | 792-952 | 424-476 |
| 18 | 9 | 48 | [a]198 | 480-565 | [d]1188-1428 | ?-2448 | ?-3960 | ?-4420 | ?-3960 | ?-2448 | 1188-1428 |
| 19 | 9 | 57 | 228 | 612-752 | 1428-1789 | ?-3876 | ?-5814 | ? [illegible] | ?-8360 | ?-5814 | ?-3876 |
| 20 | 10 | 60 | 285 | 816-912 | 2040-2506 | ?-5111 | ?-9690 | ? [illegible] | ?-16720 | ?-12920 | ?-9690 |
| 21 | 10 | 70 | 315 | 1071-1197 | 2856-3192 | ?-7518 | ?-13416 | ?-22610 | ?-27132 | ?-27132 | ?-22610 |
| 22 | 11 | 73 | 385 | 1386 | 3927-4389 | ?-10032 | ?-20674 | ?-32774 | ?-49742 | ?-54264 | ?-49742 |
| 23 | 11 | 83 | 415-419 | 1771 | 5313 | ?-14421 | ?-28842 | ? [illegible] | ?-75426 | ?-104006 | ?-104006 |
| 24 | 12 | 88 | 498 | [d]1859-2011 | [e]7084 | ?-18216 | ?-43263 | ?-76912 | ?-126799 | ?-164565 | ?-208012 |

# Distance 6 : A (n, 6, w)

| n \ w | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 6 | 1 | 2 | 1 | 1 | 1 | | | | | | |
| 7 | 1 | 2 | 2 | 1 | 1 | 1 | | | | | |
| 8 | 1 | 2 | 2 | 2 | 1 | 1 | 1 | | | | |
| 9 | 1 | 3 | 3 | 3 | 3 | 1 | 1 | 1 | | | |
| 10 | 1 | 3 | 5 | 6 | 5 | 3 | 1 | 1 | 1 | | |
| 11 | 1 | 3 | 6 | 11 | 11 | 6 | 3 | 1 | 1 | 1 | |
| 12 | 1 | 4 | 9 | [h]12[g] | 22 | 12 | 9 | 4 | 1 | 1 | 1 |
| 13 | 1 | 4 | [g]13 | [h]18[g] | [g]26 | 26 | 18 | 13 | 4 | 1 | 1 |
| 14 | 1 | 4 | 14 | [h]28 | [h]42 | [d]42-51[r] | 42 | 28 | 14 | 4 | 1 |
| 15 | 1 | 5 | 15 | [h]42 | [h]70 | [i]60-88[L] | 60-88 | 70 | 42 | 15 | 5 |
| 16 | 1 | 5 | 20 | 48 | [h]112 | [L]90-156 | [L]120-150[L] | 90-156 | 112 | 48 | 20 |
| 17 | 1 | 5 | 20[d] | [e]68 | 112-136 | 112-244[L] | 120-283 | 120-283 | 90-244 | 112-136 | 68 |
| 18 | 1 | 6 | [b]22 | 68-72 | [d]144-203[f] | 128-349 | [d]153-428[L] | ?-425[L] | 153-428 | 128-349 | 112-203 |
| 19 | 1 | 6 | [c]25-26[d] | 68-83 | [d]144-228 | 202-520[L] | ?-739 | ?-787[L] | ?-789 | ?-739 | 202-520 |
| 20 | 1 | 6 | [a]30 | [d]78-104 | 155-276 | 310-651 | ?-1199[L] | ?-1363[L] | ?-1421[L] | ?-1363 | ?-1199 |
| 21 | 1 | 7 | [a]31 | [d]102-126 | 210-364 | 465-828 | ?-1708 | ?-2364[L] | [d]1008-2702[L] | 1008-2702 | ?-2364 |
| 22 | 1 | 7 | [a]37 | [d]132-136 | 294-462 | 615-1144 | ?-2277 | ?-[illegible][L] | ?-4416[L] | [d]1288-5064[L] | ?-4416 |
| 23 | 1 | 7 | [a]40 | 133-170 | 399-521 | 969-1518 | ?-3287 | [illegible]-5819 | ?-7521[L] | ?-7953[L] | ?-7953 |
| 24 | 1 | 8 | [e]42 | [e]168-192 | [e]532-680 | [e]1368-1786 | ?-4554 | [illegible]-8770 | ?-12418[L] | ?-14632 | ?-15906[L] |

## Distance 8 : A(n, 8, w)

| n \ w | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 8 | 1 | 1 | 2 | 1 | 1 | 1 | 1 | | | | |
| 9 | 1 | 1 | 2 | 2 | 1 | 1 | 1 | 1 | | | |
| 10 | 1 | 1 | 2 | 2 | 2 | 1 | 1 | 1 | 1 | | |
| 11 | 1 | 1 | 2 | 2 | 2 | 2 | 1 | 1 | 1 | 1 | |
| 12 | 1 | 1 | 3 | 3 | 4 | 3 | 3 | 1 | 1 | 1 | 1 |
| 13 | 1 | 1 | 3 | 3 | 4 | 4 | 3 | 3 | 1 | 1 | 1 |
| 14 | 1 | 1 | 3 | 4 | 7 | 8 | 7 | 4 | 3 | 1 | 1 |
| 15 | 1 | 1 | 3 | 6 | [k] 10 | 15 | 15 | 10 | 6 | 3 | 1 |
| 16 | 1 | 1 | 4 | 6 | [k] 16 | 16-22 | 30 | 16-22 | 16 | 6 | 4 |
| 17 | 1 | 1 | 4 | 7 | [b] 17 | [j] 21-31 [L] | [m] 34-36 [L] | 34-36 | 21-31 | 17 | 7 |
| 18 | 1 | 1 | 4 | [j] 9 | [j] 20-21 | [j] 33-41 [L] | [j] 46-64 | [j] 48-72 | 46-64 | 33-41 | 20-21 |
| 19 | 1 | 1 | 4 | [j] 12 | [j] 28 | [j] 52-57 | [j] 78-97 | [j] 88-122 [L] | 88-122 | 78-97 | 52-57 |
| 20 | 1 | 1 | 5 | [j] 16 | [j] 40 | [j] 80 | [j] 130-142 | [j] 160-215 | [j] 176-244 [L] | 160-215 | 130-142 |
| 21 | 1 | 1 | 5 | [j] 21 | [j] 56 | [j] 120 | [j] 210 | [j] 280-331 | [j] 336-399 [L] | 336-399 | 280-331 |
| 22 | 1 | 1 | 5 | 21 [b] | [j] 77 | [j] 176 | [j] 330 | 280-437 [L] | [j] 616-728 | [j] 672-798 | 616-728 |
| 23 | 1 | 1 | 5 | [g] 23 | 77-80 | [j] 253 | [j] 506 | [j] 400-816 | 616-1111 [L] | [j] 1288-1417 [L] | 1288-1417 |
| 24 | 1 | 1 | 6 | [e] 24 | 77-92 | 253-274 | [j] 759 | [j] 640-[illegible] [L] | [j] 960-1639 [L] | 128[illegible]-2305 [L] | [j] 2576 |

## Distance 10 : A(n, 10, w)

| n \ w | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 1 | 1 | 1 | 2 | 1 | 1 | 1 | 1 | 1 | | |
| 11 | 1 | 1 | 1 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | |
| 12 | 1 | 1 | 1 | 2 | 2 | 2 | 1 | 1 | 1 | 1 | 1 |
| 13 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 1 | 1 | 1 | 1 |
| 14 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 1 | 1 | 1 |
| 15 | 1 | 1 | 1 | 3 | 3 | 3 | 3 | 3 | 3 | 1 | 1 |
| 16 | 1 | 1 | 1 | 3 | 3 | 3 | 4 | 3 | 3 | 3 | 1 |
| 17 | 1 | 1 | 1 | 3 | 3 | 5 | 6 | 6 | 5 | 3 | 3 |
| 18 | 1 | 1 | 1 | 3 | 4 | 6 | 9 | 10 | 9 | 6 | 4 |
| 19 | 1 | 1 | 1 | 3 | 4 | 8 | [b]12 | 19 | 19 | 12 | 8 |
| 20 | 1 | 1 | 1 | 4 | 5 | [n]10 | [b]17–18 | [g]20–24 | 38 | 20–24 | 17–18 |
| 21 | 1 | 1 | 1 | 4 | 7 | 13[b] | [g]21–26 | [g]21–41[L] | 38–49[L] | 38–49 | 21–41 |
| 22 | 1 | 1 | 1 | 4 | 7 | 15–19 | [g]22–35 | [g]22–57[L] | ?–74[L] | ?–82[L] | ?–74 |
| 23 | 1 | 1 | 1 | 4 | 8 | [b]16–23 | [g]23–50[L] | [g]2[illegible]–87[L] | ?–117[L] | ?–135[L] | ?–135 |
| 24 | 1 | 1 | 1 | 4 | 9 | [g]24–27 | [g]24–69 | ?–119[L] | ?–171[L] | ?–223[L] | ?–247[L] |

KEY TO TABLE 2

a. §4.1

b. SHEN LIN [40].

c. H.R. PHINNEY [47].

d. Miscellaneous constructions.

e. From Theorem 9 and the Steiner systems S(5,6,12), S(3,5,17), S(3,6,26), S(5,6,24), S(5,7,28), S(5,8,24) (DENNISTON [13], DOYEN & ROSA [14], WITT [66]).

f. From Theorem 6 and the nonexistence of Steiner systems S(4,5,15), S(4,6,18) (MENDELSOHN & HUNG [44], WITT [66]).

g. A cyclic code.

h. From the 3-design with $t = 3$, $v = 16$, $k = 6$, $\lambda = 4$ obtained from the Nordstrom-Robinson code (NORDSTROM & ROBINSON [46]).

i. From translates of the (16,256,6) Nordstrom-Robinson code (NORDSTROM & ROBINSON [46]).

j. From the (24,4096,8) Golay code (GOLAY [20]).

k. From translates of the (16,32,8) Reed-Muller code.

L. From linear programming.

m. From a conference matrix (SLOANE & SEIDEL [56]).

n. A quasi-cyclic code.

q. JOHNSON [31], [34].

r. W.G. VALIANT [62].

# TABLE 3

## UPPER BOUNDS FOR $T(w_1, n_1, w_2, n_2, 10)$

| | $w_2$ | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $n_2$ | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 |
| $w_1$ | $n_1$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 1 | 2 | 2* | 2* | 2* | 3* | 3* | 3* | 4* | 4* | 4* | 5* | 5* | 6* | 6* | 2* | 3* | 4* | 5* | 6 | 6 | 8 | 8 | 10 | 14 | 14 | 4* | 6 | 6 | 10 | 12 | 16 | 20 | 26 | 38 |
| 1 | 3 | 2* | 2* | 3* | 3* | 3* | 4* | 4* | 5* | 6* | 7* | 7* | 9* | | 3* | 4* | 5* | 6 | 8 | 9 | 12 | 12 | 15 | 21 | | 6* | 7 | 9 | 15 | 18 | 24 | 30 | 39 | |
| 1 | 4 | 2* | 2* | 3* | 3* | 4* | 4* | 5* | 6* | 7* | 8* | 9 | | | 4* | 4* | 5* | 7 | 9 | 12 | 16 | 16 | 20 | | | 6* | 9 | 12 | 20 | 24 | 32 | 40 | | |
| 1 | 5 | 2* | 2* | 3* | 3* | 4* | 5* | 6* | 7 | 7* | 8* | | | | 4* | 4* | 5* | 7 | 10 | 15 | 20 | 20 | | | | 6* | 10 | 15 | 24 | 30 | 40 | | | |
| 1 | 6 | 2* | 2* | 3* | 3* | 4* | 6* | 6* | 7* | 8 | | | | | 4* | 4* | 6* | 8 | 11 | 17 | 21 | | | | | 6* | 11 | 18 | 26 | 36 | | | | |
| 1 | 7 | 2* | 2* | 3* | 3* | 4* | 6* | 6* | 7* | | | | | | 4* | 4* | 7* | 8 | 11 | 17 | | | | | | 7* | 11 | 18 | 26 | | | | | |
| 1 | 8 | 2* | 2* | 3* | 3* | 4* | 6* | 6* | | | | | | | 4* | 4* | 7* | 8* | 12 | | | | | | | 8* | 11 | 18 | | | | | | |
| 1 | 9 | 2* | 2* | 3* | 3* | 4* | 6* | | | | | | | | 4* | 4* | 7* | 9* | | | | | | | | 8* | 12 | | | | | | | |
| 1 | 10 | 2* | 2* | 3* | 3* | 4* | | | | | | | | | 4* | 4* | 7* | | | | | | | | | 8* | | | | | | | | |
| 1 | 11 | 2* | 2* | 3* | 3* | | | | | | | | | | 4* | 4* | | | | | | | | | | | | | | | | | | |
| 1 | 12 | 2* | 2* | 3* | | | | | | | | | | | 4* | | | | | | | | | | | | | | | | | | | |
| 1 | 13 | 2* | 2* | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 1 | 14 | 2* | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |

| | $w_2$ | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $n_2$ | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| $w_1$ | $n_1$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 2 | 4 | 2* | 2* | 2* | 3* | 3* | 3* | 4* | 4* | 4* | 4* | 5* | 5* | 5* | 2* | 3* | 4* | 5* | 6 | 6 | 8 | 10 | 12 | 14 | 14 | 4* | 6 | 8 | 12 | 16 | 18 | 24 | 24 | 30 |
| 2 | 5 | 2* | 2* | 3* | 3* | 3* | 4* | 4* | 5* | 5* | 5* | 6* | 6* | | 3* | 4* | 5* | 6 | 8 | 10 | 12 | 15 | 17 | 20 | | 6 | 8 | 12 | 17 | 22 | 30 | 40 | 40 | |
| 2 | 6 | 2* | 3* | 3* | 3* | 4* | 4* | 5* | 6* | 6* | 7* | 9 | | | 4* | 4* | 6 | 8 | 10 | 15 | 18 | 20 | 21 | | | 6 | 11 | 15 | 21 | 30 | 45 | 60 | | |
| 2 | 7 | 2* | 3* | 3* | 4* | 4* | 5* | 7* | 7* | 8 | 9 | | | | 4* | 5* | 7* | 9 | 12 | 18 | 21 | 24 | | | | 8 | 14 | 21 | 28 | 38 | 58 | | | |
| 2 | 8 | 2* | 3* | 4* | 4* | 5* | 6* | 7* | 8 | 9 | | | | | 4* | 5* | 8 | 11 | 16 | 21 | 24 | | | | | 9 | 16 | 25 | 32 | 44 | | | | |
| 2 | 9 | 2* | 3* | 4* | 5* | 6* | 6* | 7* | 9 | | | | | | 4* | 6 | 9 | 13 | 16 | 21 | | | | | | 10 | 18 | 30 | 36 | | | | | |
| 2 | 10 | 2* | 3* | 5* | 5* | 6* | 7 | 8 | | | | | | | 5* | 6 | 10 | 15 | 19 | | | | | | | 12 | 20 | 35 | | | | | | |
| 2 | 11 | 2* | 3* | 5* | 5* | 6* | 7 | | | | | | | | 5* | 7 | 10 | 15 | | | | | | | | 12 | 21 | | | | | | | |
| 2 | 12 | 2* | 3* | 5* | 6* | 6* | | | | | | | | | 6* | 7 | 12 | | | | | | | | | 14 | | | | | | | | |
| 2 | 13 | 2* | 3* | 5* | 6* | | | | | | | | | | 6* | 7 | | | | | | | | | | | | | | | | | | |
| 2 | 14 | 2* | 3* | 5* | | | | | | | | | | | 6* | | | | | | | | | | | | | | | | | | | |
| 2 | 15 | 2* | 3* | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 2 | 16 | 2* | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |

| | $w_2$ | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $n_2$ | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
| $w_1$ | $n_1$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 3 | 6 | 2* | 2* | 2* | 3* | 3* | 3* | 4* | 4* | 4* | 4* | 4* | 4* | 4* | 2* | 3* | 4* | 4* | 6 | 6 | 7 | 8 | 9 | 10 | 12 | 4* | 6 | 8 | 11 | 16 | 20 | 24 | 30 | 34 |
| 3 | 7 | 2* | 2* | 3* | 3* | 3* | 4* | 4* | 4* | 5* | 5* | 5* | 5* | | 3* | 4* | 5* | 6 | 7 | 9 | 11 | 13 | 14 | 16 | | 6 | 8 | 12 | 18 | 23 | 33 | 41 | 46 | |
| 3 | 8 | 2* | 2* | 3* | 3* | 4* | 4* | 5* | 5* | 5* | 6* | 6* | | | 4* | 5* | 6 | 8 | 10 | 13 | 17 | 18 | 21 | | | 8* | 11 | 18 | 24 | 32 | 48 | 56 | | |
| 3 | 9 | 3* | 3* | 3* | 4* | 4* | 5* | 6* | 6* | 6* | 7* | | | | 4* | 6* | 7 | 10 | 15 | 18 | 21 | 22 | | | | 9 | 15 | 24 | 33 | 48 | 63 | | | |
| 3 | 10 | 3* | 3* | 4* | 4* | 5* | 6 | 7 | 7 | 8 | | | | | 4* | 7 | 10 | 13 | 18 | 20 | 23 | | | | | 12 | 20 | 30 | 43 | 53 | | | | |
| 3 | 11 | 3* | 4* | 4* | 5* | 6 | 7* | 8 | 9 | | | | | | 5* | 7 | 11 | 16 | 21 | 25 | | | | | | 14 | 22 | 36 | 54 | | | | | |
| 3 | 12 | 4* | 4* | 5* | 6* | 7 | 8 | 9 | | | | | | | 6* | 9 | 13 | 19 | 24 | | | | | | | 18 | 28 | 40 | | | | | | |
| 3 | 13 | 4* | 4* | 5* | 6* | 7 | 9 | | | | | | | | 6* | 9 | 15 | 19 | | | | | | | | 18 | 30 | | | | | | | |
| 3 | 14 | 4* | 5* | 5* | 6* | 8 | | | | | | | | | 6* | 10 | 15 | | | | | | | | | 20 | | | | | | | | |
| 3 | 15 | 4* | 5* | 6* | 7* | | | | | | | | | | 7 | 12 | | | | | | | | | | | | | | | | | | |
| 3 | 16 | 4* | 5* | 6* | | | | | | | | | | | 8 | | | | | | | | | | | | | | | | | | | |

* Bound is exact.

| $w_1$ | $n_1$ | $w_2$ | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 9 | 9 | 9 | 9 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $n_2$ | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 18 | 19 | 20 | 21 | 22 |
| 1 | 2 | | 8 | 12 | 18 | 24 | 40 | 52 | 70 | 20 | 38 | 52 | 82 | 114 |
| 1 | 3 | | 12 | 18 | 27 | 36 | 60 | 78 | | 30 | 57 | 78 | 123 | |
| 1 | 4 | | 16 | 24 | 36 | 48 | 80 | | | 40 | 76 | 104 | | |
| 1 | 5 | | 20 | 30 | 45 | 60 | | | | 50 | 95 | | | |
| 1 | 6 | | 22 | 36 | 54 | | | | | 60 | | | | |
| 1 | 7 | | 22 | 38 | | | | | | | | | | |
| 1 | 8 | | 22 | | | | | | | | | | | |

| $w_1$ | $n_1$ | $w_2$ | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 7 | 7 | 7 | 7 | 7 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $n_2$ | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 14 | 15 | 16 | 17 | 18 |
| 3 | 6 | | 10 | 16 | 24 | 34 | 44 | 60 | 80 | 30 | 44 | 60 | 100 | 120 |
| 3 | 7 | | 14 | 25 | 35 | 49 | 70 | 104 | | 42 | 70 | 105 | 153 | |
| 3 | 8 | | 21 | 37 | 56 | 74 | 101 | | | 56 | 100 | 148 | | |
| 3 | 9 | | 27 | 48 | 75 | 96 | | | | 84 | 129 | | | |
| 3 | 10 | | 33 | 60 | 94 | | | | | 104 | | | | |
| 3 | 11 | | 44 | 73 | | | | | | | | | | |
| 3 | 12 | | 48 | | | | | | | | | | | |

| $w_1$ | $n_1$ | $w_2$ | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 8 | 8 | 8 | 8 | 8 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $n_2$ | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 16 | 17 | 18 | 19 | 20 |
| 2 | 4 | | 10 | 14 | 18 | 30 | 36 | 48 | 60 | 24 | 36 | 54 | 72 | 120 |
| 2 | 5 | | 15 | 22 | 30 | 50 | 60 | 80 | | 40 | 60 | 90 | 120 | |
| 2 | 6 | | 18 | 30 | 45 | 72 | 90 | | | 60 | 90 | 135 | | |
| 2 | 7 | | 21 | 38 | 63 | 91 | | | | 76 | 115 | | | |
| 2 | 8 | | 28 | 44 | 72 | | | | | 88 | | | | |
| 2 | 9 | | 36 | 49 | | | | | | | | | | |
| 2 | 10 | | 40 | | | | | | | | | | | |

| $w_1$ | $n_1$ | $w_2$ | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 6 | 6 | 6 | 6 | 6 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $n_2$ | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 12 | 13 | 14 | 15 | 16 |
| 4 | 8 | | 4* | 6* | 8 | 10 | 14 | 18 | 21 | 22 | 28 | 10* | 16 | 24 | 36 | 46 | 63 | 70 | 28 | 50 | 70 | 98 | 132 |
| 4 | 9 | | 6* | 9* | 12 | 18 | 21 | 29 | 31 | 37 | | 18* | 24 | 40 | 54 | 72 | 93 | | 47 | 76 | 107 | 151 | |
| 4 | 10 | | 8 | 12 | 17 | 25 | 33 | 42 | 51 | | | 22 | 37 | 60 | 82 | 107 | | | 67 | 104 | 153 | | |
| 4 | 11 | | 10 | 18 | 25 | 35 | 48 | 55 | | | | 33 | 55 | 81 | 108 | | | | 90 | 139 | | | |
| 4 | 12 | | 14 | 21 | 33 | 48 | 63 | | | | | 42 | 66 | 102 | | | | | 116 | | | | |
| 4 | 13 | | 18 | 29 | 42 | 55 | | | | | | 58 | 89 | | | | | | | | | | |
| 4 | 14 | | 21 | 31 | 51 | | | | | | | 62 | | | | | | | | | | | |
| 5 | 10 | | | | | | | | | | | 36* | 48 | 73 | 96 | 125 | | | | | | | |
| 5 | 11 | | | | | | | | | | | 48 | 77 | 106 | 146 | | | | | | | | |
| 5 | 12 | | | | | | | | | | | 73 | 106 | 154 | | | | | | | | | |

## UPPER BOUNDS FOR $T(w_1, n_1, w_2, n_2, 10)$, cont$^{d}$.

TABLE 3 (cont)

* Bound is exact.

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